Lectures 2,3,4

Vector Analysis and Coordinate Systems

1.1 Scalar And Vector Quantities

A quantity is a scalar if it has only a magnitude at any location in space for a given time. It refers to a quantity whose value may be represented by a single (positive or negative) real number. To describe the mass of a body, all we need is the magnitude of its mass. The x, y, and z we use in basic algebra are scalars, and the quantities they represent are scalars.

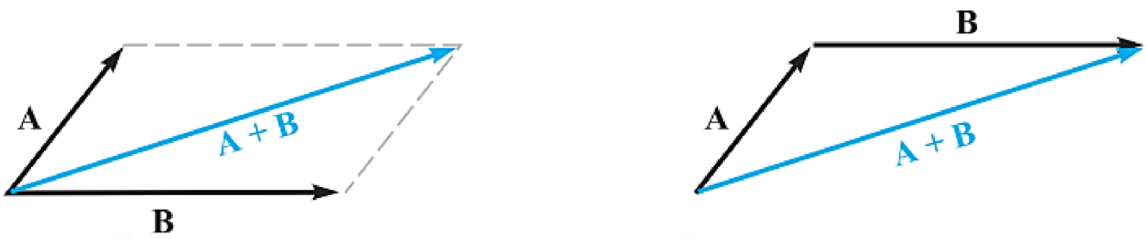
A vector quantity has both a magnitude and a direction in space. Therefore, vectors may be space and time dependent. Common vectors include displacement, velocity, force, and acceleration.

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

1.2 Vector Algebra

The addition of vectors follows the parallelogram law. Figure shows the sum of two vectors, **A** and **B**. It is easily seen that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, or that vector addition obeys the commutative law. Vector addition also obeys the associative law,

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$



The rule for the subtraction of vectors follows easily from that for addition, for we may always express $\mathbf{A} - \mathbf{B}$ as $\mathbf{A} + (-\mathbf{B})$; the sign, or direction, of the second vector is reversed, and this vector is then added to the first by the rule for vector addition.

Vectors may be multiplied by scalars. The magnitude of the vector changes, but its direction does not when the scalar is positive, although it reverses direction when multiplied by a negative scalar. Multiplication of a vector by a scalar also obeys the associative and distributive laws of algebra, leading to

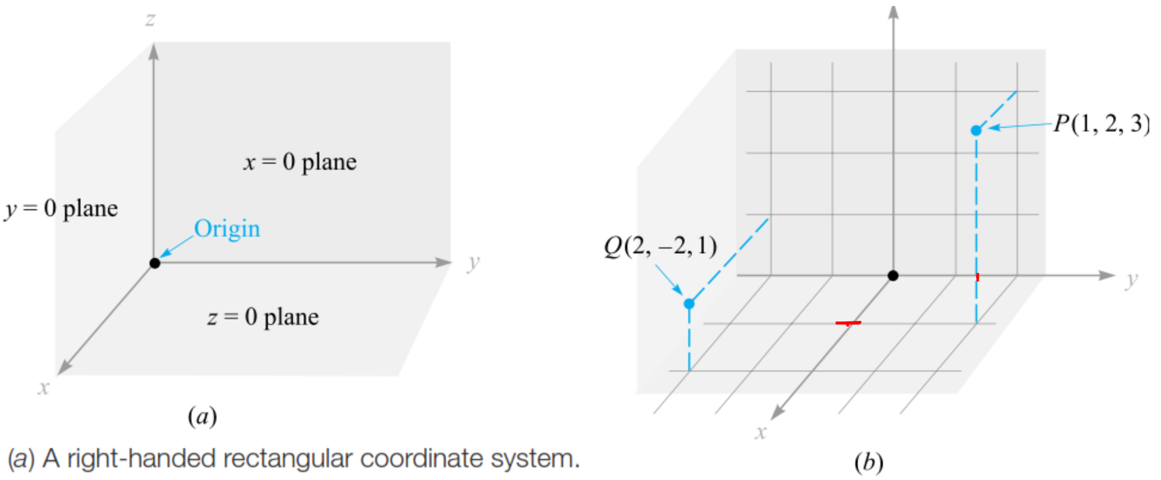
$$(r + s)(\mathbf{A} + \mathbf{B}) = r(\mathbf{A} + \mathbf{B}) + s(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B} + s\mathbf{A} + s\mathbf{B}$$

Division of a vector by a scalar is merely multiplication by the reciprocal of that scalar. Two vectors are said to be equal if their difference is zero,

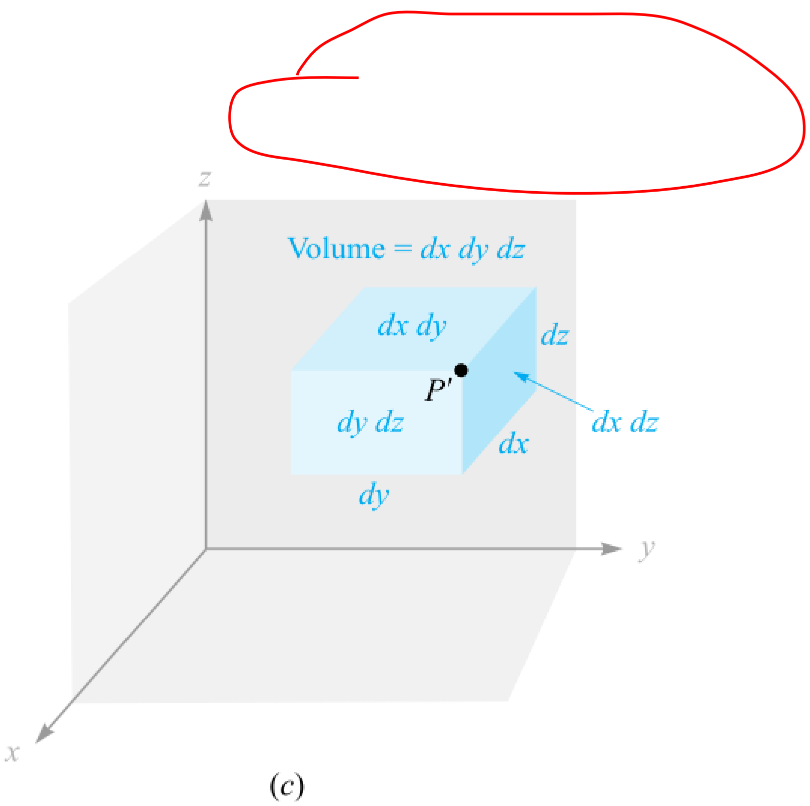
or $\mathbf{A} = \mathbf{B}$ if $\mathbf{A} - \mathbf{B} = 0$.

1.3 The Rectangular Coordinate System

In the rectangular coordinate system we set up three coordinate axes mutually at right angles to each other and call them the x, y, and z axes. It is customary to choose a right-handed coordinate system.



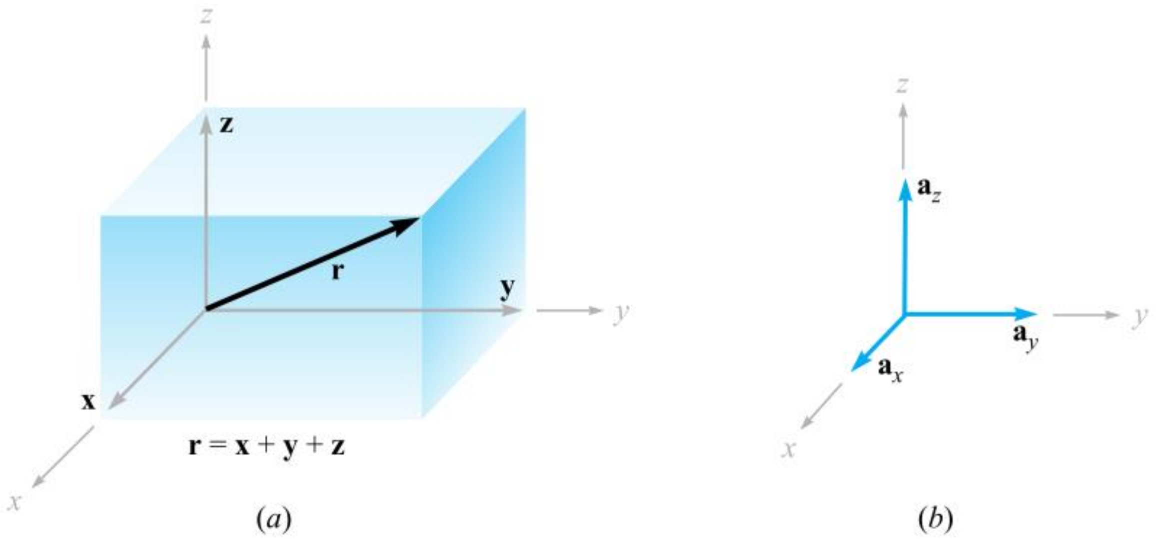
(*a*) A right-handed rectangular coordinate system.
(*b*) The location of points *P*(1, 2, 3) and *Q*(2, −2, 1).



(*c*) The differential volume element in rectangular coordinates; *dx*, *dy*, and *dz* are, in general, independent differentials.

1.4 Vector Components And Unit Vectors

The component vectors have magnitudes that depend on the given vector (such as r), but they each have a known and constant direction. This suggests the use of unit vectors having unit magnitude by definition; these are parallel to the coordinate axes and they point in the direction of increasing coordinate values. We reserve the symbol a for a unit vector and identify its direction by an appropriate subscript. Thus ax, ay and az are the unit vectors in the rectangular coordinate system. They are directed along the x, y, and z axes, respectively, as shown in Figure.



(a) The component vectors **x**, **y**, and **z** of vector **r**. (b) The unit vectors of the rectangular coordinate system have unit magnitude and aredirected toward increasing values of their respective variables.

Any vector **B** then may be described by $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The magnitude of **B** written $|\mathbf{B}|$ or simply B, is given by

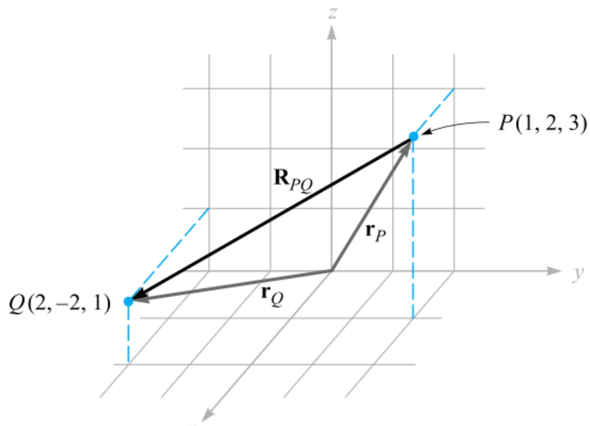
$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

and a unit vector in the direction of the vector **B** is

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

A vector r_p, a pointing from the origin to point P(1, 2, 3) is written r_p= ax + 2ay + 3az . The vector directed from P to Q may be obtained by applying the rule of vector addition. This rule shows that the vector from the origin to P plus the vector from P to Q is equal to the vector from the origin to Q. The desired vector from P(1, 2, 3) to Q(2, −2, 1) is therefore

$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$
$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$



(c)

EXAMPLE 1

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$.

Solution. We first construct the vector extending from the origin to point G,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

We continue by finding the magnitude of **G**,

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

and finally expressing the desired unit vector as the quotient,

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

A special symbol is desirable for a unit vector so that its character is immediately apparent. Symbols that have been used are $\mathbf{u}_B$, $\mathbf{a}_B$, $\mathbf{1}_\mathbf{B}$, or even **b**.

D1.1. Given points $M(-1, 2, 1)$, $N(3, -3, 0)$, and $P(-2, -3, -4)$, find: (*a*) $\mathbf{R}_{MN}$; (*b*) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$; (*c*) $|\mathbf{r}_M|$; (*d*) $\mathbf{a}_{MP}$; (*e*) $|2\mathbf{r}_P - 3\mathbf{r}_N|$.

Ans. $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$; $3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z$; 2.45; $-0.14\mathbf{a}_x - 0.7\mathbf{a}_y - 0.7\mathbf{a}_z$; 15.56

Example Three vectors are given by $\mathbf{A} = 4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$, $\mathbf{B} = 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{C} = -\mathbf{a}_x + 3\mathbf{a}_y + 6\mathbf{a}_z$, respectively. Determine the magnitude of (1) $\mathbf{R}_a = \mathbf{A} + \mathbf{B}$ and (2) $\mathbf{R}_s = \mathbf{B} - \mathbf{C}$.

Solution

1. The magnitude of $\mathbf{R}_a$ can be determined as

$$\mathbf{R}_a = 4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z + 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z = 6\mathbf{a}_x - 8\mathbf{a}_y - 3\mathbf{a}_z,$$

$$|\mathbf{R}_a| = \sqrt{6^2 + (-8)^2 + (-3)^2} = 10.44.$$

2. The magnitude of $\mathbf{R}_s$ can be calculated as

$$\mathbf{R}_s = 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z + \mathbf{a}_x - 3\mathbf{a}_y - 6\mathbf{a}_z = 3\mathbf{a}_x - 8\mathbf{a}_y - 10\mathbf{a}_z,$$

$$|\mathbf{R}_s| = \sqrt{3^2 + (-8)^2 + (-10)^2} = 13.15.$$

Example A unit vector is parallel to the resultant (addition) vector of $\mathbf{A} = 2\mathbf{a}_x + 3\mathbf{a}_y + 6\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x - \mathbf{a}_y - 2\mathbf{a}_z$. Determine the unit vector.

Solution The resultant vector can be determined as

$$\mathbf{R}_a = \mathbf{A} + \mathbf{B} = 7\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z.$$

The unit vector can be calculated as

$$\mathbf{a}_u = \frac{\mathbf{R}_a}{|\mathbf{R}_a|} = \frac{7\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{7^2 + 2^2 + 4^2}} = 0.84\mathbf{a}_x + 0.24\mathbf{a}_y + 0.48\mathbf{a}_z.$$

Differential displacement is given by $dL = dx a_x + dy a_y + dz a_z$

$d_L = |d_L| = \sqrt{dx^2 + dy^2 + dz^2}$

1. **Differential volume is given by** $dv = dxdydz$
2. **Differential normal area is given by** $ds = dydz\, a_x = dxdz\, a_y = dxdy\, a_z$

Example: express the unit vector directed toward the point p (1,-2, 3) from an arbitrary point on the line described by x=-2, y= 2.

Point a(-2,-2,z) is an arbitrary point on the line x=-2, y= 2. Then Rap = $3a_x + (-2-2)a_y + (3-z)a_z = 3a_x - 4a_y + (3-z)a_z$

$$aRap = Rap/|Rap| = \frac{3ax - 4ay + (3-z)az}{\sqrt{9+16+(3-z)^2}} = \frac{3ax - 4ay + (3-z)az}{\sqrt{25+(3-z)^2}}$$

Example : find the length of the curve z=y=x^2 from A(0,0,0) to B(1,1,1)

L= $\int_A^B dL = \int_A^B \sqrt{dx^2 + dy^2 + dz^2} = \int_{x=0}^{x=1} \sqrt{1 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2 dx}$

dy/dx= 2x; dz/dx=2x ; L= $\int_{x=0}^{x=1} \sqrt{1 + 4(x)^2 + 4(x)^2 dx} = \int_0^1 1 + 8x^2 dx$

let x=$\frac{1}{\sqrt{8}} sinhy = dx = \frac{1}{\sqrt{8}} coshydy$; $\sqrt{1 + 8x^2} = \sqrt{1 + sinh^2 y} = coshy$

$$L=\int_{x=0}^{1} \frac{1}{\sqrt{8}} cosh^2 ydy = \frac{1}{\sqrt{8}} \int_0^1 \frac{1}{2}(1 + cosh2y)dy$$

$\frac{1}{2\sqrt{8}} y + \frac{1}{4\sqrt{8}} sinh2y \mid_0^1 = \frac{1}{2\sqrt{8}} sinh^{-1}\sqrt{8}x + \frac{2sinhy\ coshy}{4\sqrt{8}} \mid_0^1$

Sinhy=$\sqrt{8}x$; -$sinh^2$y+$cosh^2$y=1

coshy= $\sqrt{1 + sinh^2 y} = \sqrt{1 + 8x^2}$

L= $\frac{1}{2\sqrt{8}} sinh^{-1}\sqrt{8}x + \frac{x\sqrt{1+8x^2}}{2} \mid_0^1$; L= $\frac{1}{2\sqrt{8}} sinh^{-1}\sqrt{8}x + \frac{3}{2}$ = 1.8116(m)

1.5 The Vector Field

A vector field defined as a vector function of a position vector. In general, the magnitude and direction of the function will change as we move throughout the region, and the value of the vector function must be determined using the coordinate values of the point in question. the velocity field of the particle in space ,electric field intensity, magnetic field are examples of vector field.

Example :A vector field is given by F=4x^2y ax – (7x+2z) ay (4xy+2z^2) az

(a) Evaluate |F| at p(2,-3,4); (b) find a unit vector specifying the direction of F at p; (c) describe the locus of all points on the z- axis for which |F| =1

Solution:

(a) at p ; F -48 ax -22 ay + 8 az; |F| = 53.4
(b) a_F = F/|F| = -0.8988 ax – 0.4119 ay + 0.1498 az
(c) on z- axis ,x=0, y=0, then F = -2z ay + 2z^2 az
    |F| =1= $\sqrt{4z^2 + 4z^4}$ ≫ z= ∓ 0.455 (two points)

1.6 The Dot Product

Given two vectors **A** and **B**, the *dot product*, or *scalar product*, is defined as

$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \qquad \mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z.$

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta_{AB}$$

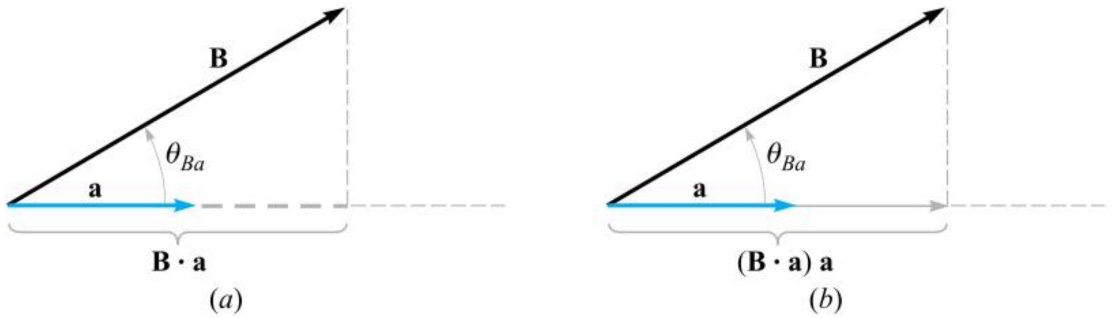
$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\mathbf{A}\cdot\mathbf{A} = A^2 = |\mathbf{A}|^2$$

$\mathbf{a}_A\cdot\mathbf{a}_A = 1,$ any unit vector dotted with itself is unity

$$\mathbf{B}\cdot\mathbf{a} = |\mathbf{B}|\,|\mathbf{a}|\cos\theta_{Ba} = |\mathbf{B}|\cos\theta_{Ba}$$

Thus, $\mathbf{B}\cdot\mathbf{a}$ is the projection of **B** in the **a** direction.



projection of **B** (*a*) The scalar component of **B** in the direction of the unit vector **a** is **B** · **a**. (*b*) The vector component of **B** in the direction of the unit vector **a** is (**B** · **a**)**a**.

EXAMPLE In order to illustrate these definitions and operations, consider the vector field $\mathbf{G} = y\mathbf{a}_x - 2.5x\mathbf{a}_y + 3\mathbf{a}_z$ and the point $Q(4, 5, 2)$. We wish to find: **G** at Q; the scalar component of **G** at Q in the direction of $\mathbf{a}_N = \frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$; the vector component of **G** at Q in the direction of $\mathbf{a}_N$; and finally, the angle θ_{Ga} between $\mathbf{G}(\mathbf{r}_Q)$ and $\mathbf{a}_N$.

Solution. Substituting the coordinates of point Q into the expression for **G**, we have

$$\mathbf{G}(\mathbf{r}_Q) = 5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z$$

Next we find the scalar component. Using the dot product, we have

$$\mathbf{G}\cdot\mathbf{a}_N = (5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z)\cdot\frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = \frac{1}{3}(10 - 10 - 6) = -2$$

The vector component is obtained by multiplying the scalar component by the unit vector in the direction of $\mathbf{a}_N$,

$$(\mathbf{G}\cdot\mathbf{a}_N)\mathbf{a}_N = -(2)\tfrac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = -1.333\mathbf{a}_x - 0.667\mathbf{a}_y + 1.333\mathbf{a}_z$$

The angle between $\mathbf{G}(\mathbf{r}_Q)$ and $\mathbf{a}_N$ is found from

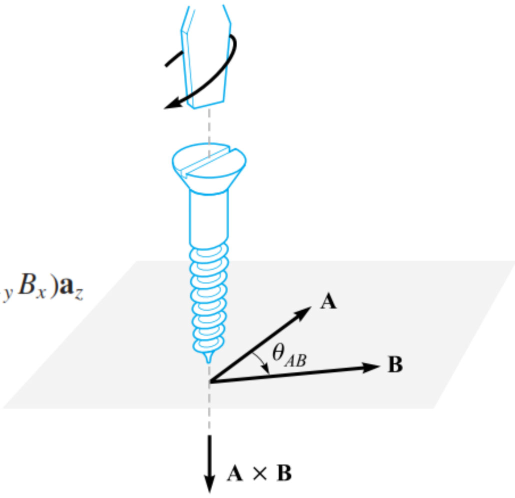
$$\mathbf{G}\cdot\mathbf{a}_N = |\mathbf{G}|\cos\theta_{Ga} = -2 = \sqrt{25+100+9}\cos\theta_{Ga} \text{ and } \theta_{Ga} = \cos^{-1}\frac{-2}{\sqrt{134}} = 99.9^\circ$$

1.7 The Cross Product

The cross product $\mathbf{A}\times\mathbf{B}$ is a vector; the magnitude of $\mathbf{A}\times\mathbf{B}$ is equal to the product of the magnitudes of $\mathbf{A}$, $\mathbf{B}$, and the sine of the smaller angle between $\mathbf{A}$ and $\mathbf{B}$; the direction of $\mathbf{A}\times\mathbf{B}$ is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A}\times\mathbf{B} = \mathbf{a}_N|\mathbf{A}|\,|\mathbf{B}|\sin\theta_{AB}$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



$\mathbf{a}_x\times\mathbf{a}_y = \mathbf{a}_z$, $\mathbf{a}_y\times\mathbf{a}_z = \mathbf{a}_x$, and $\mathbf{a}_z\times\mathbf{a}_x = \mathbf{a}_y$.

$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z$

$\mathbf{A}\times\mathbf{A} = 0$

$(\mathbf{A}\times\mathbf{B}) \neq (\mathbf{B}\times\mathbf{A})$.

$(\mathbf{A}\times\mathbf{B}) = -(\mathbf{B}\times\mathbf{A})$.

$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$.

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$.

Thus, if $\mathbf{A} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = -4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$, we have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ -4 & -2 & 5 \end{vmatrix}$$

$$= [(-3)(5) - (1(-2)]\mathbf{a}_x - [(2)(5) - (1)(-4)]\mathbf{a}_y + [(2)(-2) - (-3)(-4)]\mathbf{a}_z$$

$$= -13\mathbf{a}_x - 14\mathbf{a}_y - 16\mathbf{a}_z$$

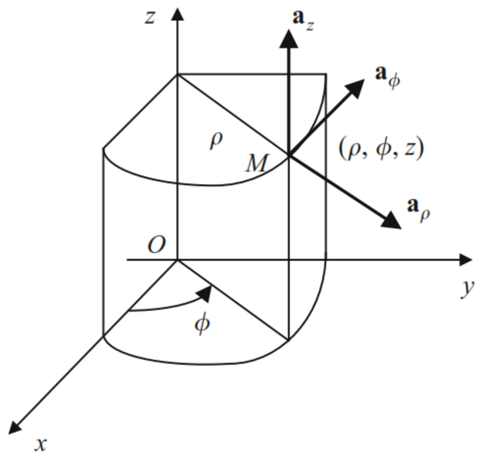
Example Three vectors are given by $\mathbf{A} = 2\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z$, $\mathbf{B} = \mathbf{a}_x + 3\mathbf{a}_y - 5\mathbf{a}_z$ and $\mathbf{C} = 3\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z$, respectively. Determine the vector $\mathbf{A}\bullet\mathbf{B}\times\mathbf{C}$.

Solution (1) The cross product can be determined as

$$\mathbf{A}\bullet\mathbf{B}\times\mathbf{C} = \begin{vmatrix} 2 & 3 & -5 \\ 1 & 3 & -5 \\ 3 & 4 & -6 \end{vmatrix} = 2(-18+20) - 3(-6+15) - 5(4-9) = 2.$$

1.8 Circular Cylindrical Coordinates

The circular cylindrical coordinate system is normally known as the cylindrical coordinate system. The cylindrical coordinate usually refers to the three-dimensional polar coordinate in analytical geometry. This coordinate is represented by (ρ, φ, z). The ρ coordinate represents the radius of the cylinder, φ represents the magnitude of the circumference of the specific point on the surface of the cylinder and z represents the coordinate as represented by rectangular coordinate system. Consider a point M(ρ, φ, z) on the cylindrical system as shown in Fig .The vector A can be written in terms of its components as



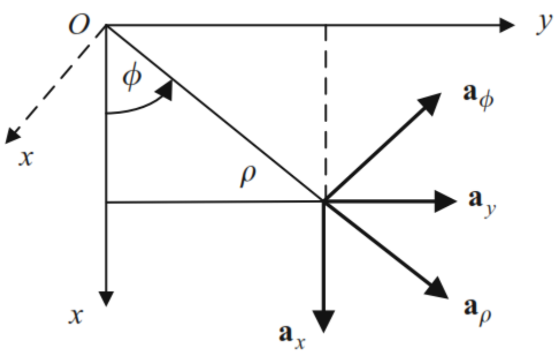
$$\mathbf{A} = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z \equiv (A_\rho, A_\phi, A_z).$$

The ranges of the coordinates are

$$0 < \rho < \infty,\ 0 < \phi < 2\pi,\ -\infty < z < \infty.$$

The magnitude of the cylindrical vector is

$$|\mathbf{A}| = \sqrt{A_\rho{}^2 + A_\phi{}^2 + A_z{}^2}.$$



the properties of unit vectors are

$$\mathbf{a}_\rho \bullet \mathbf{a}_\rho = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = \mathbf{a}_z \bullet \mathbf{a}_z = 1,$$

$$\mathbf{a}_\rho \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_z = \mathbf{a}_z \bullet \mathbf{a}_\rho = 0,$$

$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z, \qquad \mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho \qquad \mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi.$$

the relations from Cartesian to cylindrical coordinates can be written as

$$\rho = \sqrt{x^2 + y^2},$$

$$\phi = \tan^{-1}\frac{y}{x}$$

the relations from cylindrical to Cartesian coordinates can be written

$$x = \rho\cos\phi,$$

$$y = \rho sin\phi$$

$$z = z.$$

the conversion of unit vectors from cylindrical to Cartesian is

$$\mathbf{a}_x = \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi,$$

$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \mathbf{a}_z.$$

The dot products of unit vectors in Cartesian and cylindrical coordinate systems can be determined as

$$A_x = \mathbf{A}\bullet\mathbf{a}_x = A_\rho\cos\phi - A_\phi\sin\phi,$$

$$A_y = \mathbf{A}\bullet\mathbf{a}_y = A_\rho\sin\phi + A_\phi\cos\phi$$

$$A_z = \mathbf{A}\bullet\mathbf{a}_z = A_z.$$

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\bullet$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y\bullet$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\bullet$	0	0	1

$$A_\rho = \mathbf{A}\bullet\mathbf{a}_\rho = A_x\cos\phi + A_y\sin\phi,$$

$$A_\phi = \mathbf{A}\bullet\mathbf{a}_\phi = -A_x\sin\phi + A_y\cos\phi$$

$$A_z = \mathbf{A}\bullet\mathbf{a}_z = A_z$$

the differential length, can be written as

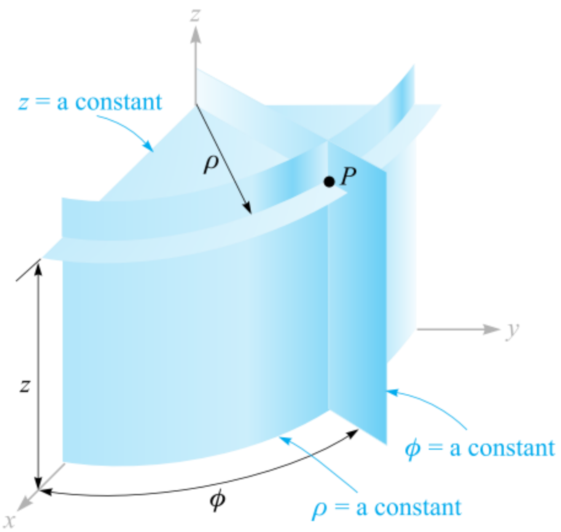
$$d\mathbf{l} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z,$$

normal surface

$$d\mathbf{S} = \rho\,d\phi\,dz\,\mathbf{a}_\rho,$$
$$= d\rho\,d\phi\,\mathbf{a}_\phi,$$
$$= \rho\,d\rho\,d\phi\,\mathbf{a}_z$$

and volume

$$dv = \rho\,d\rho\,d\phi\,dz.$$



EXAMPLE Transform the vector $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$ into cylindrical coordinates.

Solution. The new components are

$$B_\rho = \mathbf{B}\cdot\mathbf{a}_\rho = y(\mathbf{a}_x\cdot\mathbf{a}_\rho) - x(\mathbf{a}_y\cdot\mathbf{a}_\rho)$$
$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$
$$B_\phi = \mathbf{B}\cdot\mathbf{a}_\phi = y(\mathbf{a}_x\cdot\mathbf{a}_\phi) - x(\mathbf{a}_y\cdot\mathbf{a}_\phi)$$
$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

Thus, $\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z$

1.9 The Spherical Coordinate System

The point in spherical coordinates is defined as $M(r,\ \theta,\phi)$. A radial line with the length of r is drawn at an angle θ with the z-axis and the unit vectors of this system are $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ as shown in Fig. Here, $\mathbf{a}_r$ is parallel to the radial line and the unit vector $\mathbf{a}_\phi$ is tangent to the sphere, and it increases in the direction of increasing ϕ. The unit vector $\mathbf{a}_\theta$ is basically a tangent to the sphere, and it increases in the direction of increasing θ. The vector $\mathbf{A}$ in terms of spherical components can be written as

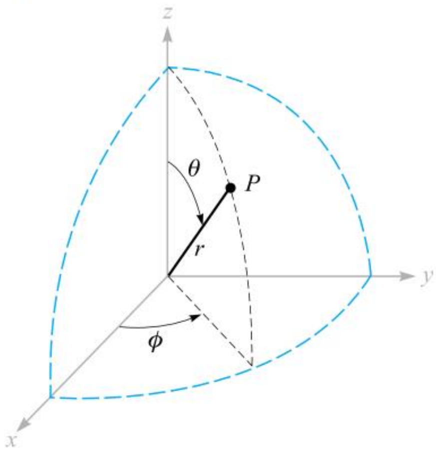
$$\mathbf{A} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi \equiv (A_r, A_\theta, A_\phi).$$

The ranges of the coordinates are $0 < r < \infty,\ 0 < \theta < \pi,\ 0 < \phi < 2\pi.$

The magnitude of the vector can be written as $|\mathbf{A}| = \sqrt{A_r{}^2 + A_\theta{}^2 + A_\phi{}^2}.$

The transformation of scalars from the rectangular to the spherical coordinate

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$



The transformation in the reverse direction

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

Dot products of unit vectors in spherical and rectangular coordinate systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x\cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y\cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z\cdot$	$\cos\theta$	$-\sin\theta$	0

In this coordinate system, the properties of unit vectors are

$$\mathbf{a}_r \bullet \mathbf{a}_r = \mathbf{a}_\theta \bullet \mathbf{a}_\theta = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = 1,$$

$$\mathbf{a}_r \bullet \mathbf{a}_\theta = \mathbf{a}_\theta \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_r = 0,$$

$$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi, \qquad \mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r\ ,\ \mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta.$$

the following expressions can be written

$\rho = r\sin\theta,\quad x = \rho\cos\phi,\ y = \rho\sin\phi \qquad z = r\cos\theta.$

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi.$

the relationship between the Cartesian and the spherical coordinates.

$$r = \sqrt{x^2 + y^2 + z^2}\ ,\quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right),\quad \phi = \tan^{-1}\left(\frac{y}{x}\right).$$

$$\mathbf{a}_r = \sin\theta\cos\phi\mathbf{a}_x + \sin\theta\sin\phi\mathbf{a}_y + \cos\theta\mathbf{a}_z.$$
$$\mathbf{a}_\theta = \cos\theta\cos\phi\mathbf{a}_x + \cos\theta\sin\phi\mathbf{a}_y - \sin\theta\mathbf{a}_z.$$
$$\mathbf{a}_\phi = -\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y.$$

The position vector in Cartesian coordinates is

$$\mathbf{r} = r\sin\theta\cos\phi\mathbf{a}_x + r\sin\theta\sin\phi\mathbf{a}_y + r\cos\theta\mathbf{a}_z.$$

$$A_r = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta,$$

$$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$$
$$A_\phi = -A_r\sin\phi + A_\theta\cos\phi.$$

$$A_x = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi,$$

$$A_y = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$

$$A_z = A_r\cos\theta - A_\theta\sin\theta.$$

The expression of differential length is $d\mathbf{l} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi.$

the expressions of differential area

$$d\mathbf{S} = r^2\sin\theta d\theta d\phi\mathbf{a}_r,$$
$$= r\sin\theta dr d\phi\mathbf{a}_\theta,$$
$$= r dr d\theta\mathbf{a}_\phi,$$

and volume can be written as $dv = r^2\sin\theta dr d\theta d\phi.$

Example A point in Cartesian coordinates is given by $P(1,3,2)$. Convert this point into spherical coordinates.

Solution The components of spherical coordinates can be determined as

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+3^2+2^2} = 3.74,$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 57.69^\circ \text{ and}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{1}\right) = 71.57^\circ.$$